A robust control method of a hybrid vibration absorber for vibration suppression in a wide frequency band considering large structural parameter variations

Kai Yang¹, Long Cui² and Hai Huang¹

Abstract
This article presents a robust control method of an active–passive hybrid vibration absorber for vibration suppression in a wide frequency band based on skyhook damping strategy. The technology of linear extended state observer is applied to automatically change pole locations of the system for stabilising the skyhook damping control, so that a high control gain can be used for effective control performance. Simulations on the vibration control of a single-degree-of-freedom primary structure via a hybrid vibration absorber are conducted to analyse the stability and performance of the present method. The results reveal that the present method is effective for vibration suppression and robust against marked parameter variations in both the primary structure and the absorber, when the controller is stabilised. Finally, this article demonstrates the effectiveness of the present approach for active control of multiple hybrid vibration absorbers to suppress multimode vibration. Simulation results show that the robustness against structural parameter variations is significantly improved compared to the previous approach, and all the vibration modes of the primary structure are suppressed effectively. In addition, by means of commercial dynamics software, simulations on vibration control of a cantilever panel via two hybrid absorbers are performed to further demonstrate the control effectiveness for a complex structure, where an analytical model is not available. Results indicate that the present method is effective and stable despite the large changes of the parameters.

Keywords
Hybrid vibration absorber, active vibration control, linear extended state observer, multimode vibration control, parameter variation

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Introduction
Vibration control of flexible structures is highly concerned in recent decades.¹ Although a variety of methods by embedding smart materials in the structures have been proposed for damping enhancement,² they are not well suited for those low-strain structures that are not practicable to be embedded with the damping materials.³

A dynamic vibration absorber (DVA) is an effective device for vibration control of those structures that are infeasible to be embedded with smart materials.³ With the advantages of reliability, efficiency, and low maintenance cost,⁴ DVA is an ideal damping device applied in aerospace and civil engineering.³,⁵,⁶ Since a passive DVA is only effective to suppress the resonant vibration when it is pre-tuned to the natural frequency of the primary structure,⁷ an active–passive hybrid DVA, including an acting force, has been proposed to improve the performance and widen the bandwidth.

Most previous active control methods of a hybrid DVA are designed based on models of primary structures, such as assignment of zeros and poles,⁸ band

¹Department of Spacecraft Technology, School of Astronautics, Beihang University, Beijing, People’s Republic of China
²State Key Laboratory of Robotics, Shenyang Institute of Automation, Chinese Academy of Sciences, Shenyang, People’s Republic of China

Corresponding author:
Hai Huang, School of Astronautics, Beihang University, No. 37, Xueyuan Road, Beijing 100191, People’s Republic of China.
Email: hhuang@buaa.edu.cn
pass filter,\textsuperscript{9} acceleration feedback using a time delay method,\textsuperscript{10} feedforward scheme,\textsuperscript{11} and linear quadratic Gaussian (LQG) active control,\textsuperscript{12} thus they are unable to retain control performance and stability for those situations where uncertainties and parameter variations in structures are significant.\textsuperscript{13,14} To solve this problem, control approaches that are robust against the uncertainties and variations have been proposed by Utsumi,\textsuperscript{15} Wu,\textsuperscript{16} Wu et al.,\textsuperscript{17} Morgan and Wang,\textsuperscript{18} and Hillis.\textsuperscript{18} For example, Wu\textsuperscript{15} and Wu et al.\textsuperscript{16} proposed a virtual vibration absorber method by using an active control force to emulate an additional passive vibration absorber. Morgan and Wang\textsuperscript{17} presented a piezoelectric absorber with an adjustable shunt circuit to suppress harmonic vibrations with varying frequencies. Since these methods can only suppress the tonal vibration with specific frequencies by tuning the DVA parameters actively, they are not well suited for vibration control in a wide bandwidth.\textsuperscript{13} For vibration control in a wide frequency band, Utsumi\textsuperscript{15} proposed an active control method based on the skyhook damping strategy. This study by Utsumi investigated that the natural frequency of a hybrid DVA can destabilise the closed-loop system when a high control gain for the skyhook damping is utilised, thus a stabilisation method was designed by introducing the value of the DVA stiffness into the controller directly. According to the results in the study by Utsumi, this method needs to identify the accurate DVA natural frequency, thus it is not robust to the significant variations in this parameter. Hillis\textsuperscript{18} developed a control strategy to accommodate an immeasurable wave disturbance force by incorporating an extended state observer. Despite its robustness against marked parameter variations in the primary structure, it can be destabilised due to minor changes in the DVA parameters as well, and the request for improving the robustness to parameter variations in the hybrid DVA is mentioned in the conclusion. Hence, a control method of a hybrid DVA for vibration suppression in a wide frequency band, which is strongly robust against significant parameter variations in both primary structures and hybrid DVAs, is demanded for application.

A new active control method of a hybrid DVA, robust against significant structural parameter variations, is proposed in this article for vibration suppression in a wide frequency band. This method combines the skyhook damping strategy, which is robust and effective for vibration suppression,\textsuperscript{19} and the technology of linear extended state observer (LESO)\textsuperscript{20} to solve the robustness problem of the previous approaches and achieve effective vibration control performance. According to the theoretical analysis in this article, direct application of skyhook damping strategy is unable to achieve effective performance due to the limitation of the natural frequency of the hybrid DVA. To solve the problem, the technology of LESO is utilised to automatically change pole locations of the hybrid DVA system, so that the control gain used in the skyhook damping control is not limited by the DVA natural frequency. A single-degree-of-freedom (SDOF) primary structure with a hybrid DVA is used as an example to analyse the robustness and performance of the present method. For comparison with a previous approach, numerical simulations on the vibration suppression of a mass–spring layered structure via two hybrid DVAs are conducted to verify the multimode vibration control performance. Finally, to further validate the vibration control effectiveness for complex multimode structures, where analytical models are not available, simulations on vibration control of a cantilever panel are performed based on ADAMS–Simulink co-simulation approach.\textsuperscript{21}

Theory

Skyhook damping strategy

The SDOF primary structure with a hybrid DVA is shown in Figure 1. The mass and the stiffness of the primary structure are \( m \) and \( K \), respectively; the hybrid DVA is also an SDOF system with a mass \( m_a \), a stiffness \( k_a \), a damping \( c_a \), and an active control force \( f_c \).

The dynamic equations of the system subjected to disturbance \( f_d \) are presented as follows:

\[
\begin{align*}
\begin{cases}
M(\ddot{x}_1 + \Omega^2 x_1) &= f_d - P, \\
P &= \mu M(\ddot{u}_a + \dot{x}_1) - f_c = \mu M(2\xi \omega_o \dot{u}_a + \omega_o^2 u_a)
\end{cases}
\end{align*}
\]

where \( x_1 \) and \( u_a \) represent the translation of the primary structural mass and the relative motion of the DVA mass, respectively; \( \Omega \) and \( \omega_o \) are the natural frequency of the primary structure and the DVA natural frequency, respectively; \( \xi \) is the damping ratio of the DVA; \( \mu = m_a/M \); and \( P \) is the force acting on the primary structure generated by the DVA. When \( f_c = 0 \) and \( 1/(1 + \mu) \approx 1 \), the absorber can be tuned by \( \omega_a = \Omega \) to obtain the optimal passive vibration control performance for suppressing the resonance of the primary structure.\textsuperscript{22}

To improve the control performance and widen the bandwidth, skyhook damping strategy is employed, that is, \( f_c = Q \ddot{x}_1, Q > 0 \), thus the transfer equation from the excitation \( f_d \) to \( \ddot{x}_1 \) is presented to be

\[
T(s) = \frac{\ddot{x}_1(s)}{F_d(s)} = \frac{s^2(s^2 + 2\xi \omega_o s + \omega_o^2)}{M(a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4)}
\]

Figure 1. An SDOF structure with a hybrid DVA.
DVA: dynamic vibration absorber.
where $a_0 = 1$, $a_1 = Q/M + 2\xi \omega_o (1 + \mu)$, $a_2 = (1 + \mu) \omega_o^2 + \Omega^2$, $a_3 = 2\xi \omega_o \Omega^2$, and $a_4 = \Omega^2 \omega_o^2$: $Q$ should be chosen to be as large as possible in order to ensure a notable performance because

$$\frac{\partial |\mathbf{J}(\omega)|}{\partial Q} = \frac{\partial |\mathbf{J}(\omega)|}{\partial a_1} \frac{\partial a_1}{\partial Q}
= -\omega_o^5 \frac{2(a_1 \omega_o^3 - a_2 \omega_o) \sqrt{(\omega_o - \omega)^2 + 4\xi^2 \omega_o^2 \omega^2}}{M^2 \left[(a_0 \omega_o^4 - a_2 \omega^2 + a_3 \omega_o^2) \right]^2}$$

(3)

As is shown in equation (3), in the frequency band where $\omega > \sqrt{a_3/a_1}$, $|\mathbf{J}(\omega)|$ can be reduced by increasing $Q$. In addition, $\sqrt{a_3/a_1}$ is decreased by choosing a large $Q$, since $a_1 = Q/M + 2\xi \omega_o (1 + \mu)$. Hence, the frequency bandwidth of the active control, where $\frac{\partial |\mathbf{J}(\omega)|}{\partial Q} < 0$, can be increased as well.

According to the Routh–Hurwitz criterion, to ensure the stability of the closed-loop system, $a_{ii}, i = 1, \ldots, 4$ should satisfy the following inequalities

$$\begin{cases}
a_0 > 0, a_1 > 0, a_4 > 0 \\
D_2 = a_1 a_2 - a_0 a_3 > 0 \\
D_3 = a_2 D_2 - a_0 a_4 > 0
\end{cases}$$

(4)

In this case, $D_2 > 0$ regardless of the systematic parameters. As is mentioned before, for optimal passive vibration suppression, $\omega_o = \Omega$; thus, to ensure $D_3 > 0$, $Q$ must satisfy

$$Q < \left( \sqrt{\mu^2 + 4\mu - \mu} \right) M \xi \Omega$$

(5)

Hence, a high control gain cannot be applied to achieve effective vibration control performance due to the small value of the primary natural frequency. Utsumi proposed a stabilisation approach to transform the control law by feeding $u_0$ back to the control loop with a positive gain $k_0$, where $k_0$ must equal $m_o \omega_o^2$ accurately, and then $D_3 > 0$ despite arbitrary systematic parameters. However, this stabilisation approach is too ideal to be applied because of the uncertainties and parameter variations in the DVA stiffness. Moreover, if minor decrease of the DVA natural frequency causes $k_0 > m_o \omega_o^2$, the DVA becomes a negative stiffness system that is unstable. Hence, a stabilisation method that is strongly robust to the parameter variations in the hybrid DVA is demanded.

**Stabilisation approach**

Add a compensation force $f_c$ into the control law, and then the control force becomes

$$f_c = Q\ddot{x}_1 + f_c$$

(6)

The dynamic equation of the DVA can be depicted as

$$m_o \ddot{u}_0 = Q\ddot{x}_1 + f_c - m_o \ddot{x}_1 - 2m_o \ddot{u}_0 - 2m_o (\xi \omega_o - \alpha) \dot{u}_0 - m_o \omega_o^2 u_0$$

(7)

where $\alpha$ is a positive constant. Defining $d = -2(\xi \omega_o - \alpha) \dot{u}_0 - \omega_o^2 u_0$, which represents the perturbation causing the instability of the closed-loop system; defining $F = Q\ddot{x}_1 + f_c - m_o \ddot{x}_1$ and assuming $d = h$, equation (7) can be written as

$$\ddot{z} = A \ddot{x} + BF + Eh$$

(8)

where

$$z = \begin{bmatrix} u_a \\ u_0 \\ d \\ \end{bmatrix}, 
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2\alpha & 1 \\ 0 & 0 & 0 \end{bmatrix}, 
B = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, 
E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 
b = \frac{1}{m_o}$$

(9)

To estimate $d$, establish the following LESO

$$\ddot{z} = A \ddot{x} + BF + L(u_o - \ddot{z})$$

(10)

where $L = [l_1, l_2, l_3]^T$, which is the observer gain vector. By subtracting equation (10) from equation (8), the state error equation is written as follows

$$\ddot{e} = K e + Eh$$

(11)

where $e = z - \ddot{z}$, which is the vector representing for the state estimation errors. The matrix $K$ is presented to be

$$K = \begin{bmatrix} -l_1 & 1 & 0 \\ -l_2 & -2\alpha & 1 \\ -l_3 & 0 & 0 \end{bmatrix}$$

(12)

If all the eigenvalues of $K$ are negative, the state estimation errors can tend to zero. Therefore, $l_1$, $l_2$, and $l_3$ are chosen to be as follows

$$\begin{cases}
l_1 = 3l_0 - 2\alpha \\
l_2 = 3l_0^2 - 6l_0\alpha + 4\alpha^2, \ l_0 > 0 \\
l_3 = l_0^3
\end{cases}$$

(13)

where $l_0$ is the observer gain that adjusts all the three eigenvalues of $K$ to be $-l_0$. By increasing the value of $l_0$, the state estimation errors can be increasingly reduced. When $\ddot{z}_3$ (the estimation value of $d$) is observed by equation (10), let $f_c = -(1/b)\ddot{z}_3$, and then equation (7) is approximately transformed to be

$$m_o \ddot{u}_0 + 2m_o \ddot{u}_0 + 2m_o \ddot{u}_0 \approx Q\ddot{x}_1 - m_o \ddot{x}_1$$

(14)

The characteristic equation of the closed-loop system becomes

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

(15)

where $a_0 = 1$, $a_1 = 2(1 + \mu)\alpha + Q/M$, $a_2 = \Omega^2$, and $a_3 = 2\alpha \Omega^2$.

It is obvious that the stability is ensured when it is stabilised, because

$$\begin{cases}
a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0 \\
a_1 a_2 - a_0 a_3 = \frac{4\alpha^2 (\mu + 1)}{m_o Q^2} > 0
\end{cases}$$
The square of the magnitude transmissibility from $f_d$ to $\ddot{x}_1$ is

$$|T(\omega)|^2 \approx \frac{\omega^4}{M^2} \left( 4\alpha^2 + \omega^2 - \frac{a_0^2 \omega^6 + (a_1^2 - 2a_0a_2)\omega^4 + (a_2^2 - 2a_1a_3)\omega^2 + a_3^2}{\alpha^2} \right)$$

(16)

If $\omega \geq \sqrt{a_3/a_1}$

$$\frac{\partial |T(\omega)|^2}{\partial Q} \approx \frac{2\omega(4\alpha^2 + \omega^2)(\alpha^2 - \alpha)}{M^2[\alpha^2 \omega^6 + (a_1^2 - 2a_0a_2)\omega^4 + (a_2^2 - 2a_1a_3)\omega^2 + a_3^2]} < 0$$

(17)

As is shown in inequality (17), lower bound of the bandwidth where effective control performance can be achieved is determined by the value of $a_3/a_1$. In order to widen the bandwidth, that is, decreasing the value of $a_3/a_1$, increase $Q$ and decrease $\alpha$. However, when the system is stabilised, $\alpha$ cannot be zero, because

$$U_0(s) \approx \frac{Q}{m_0(s + 2\alpha)} X_1(s)$$

(18)

If $s \rightarrow 0$, the motion of the DVA mass can be finite due to a non-zero $\alpha$.

The design procedure of this stabilisation approach is given as follows:

**Step 1.** Measure the absorber mass $m_a$, choose a control gain $Q$ for skyhook damping, and choose a positive constant $\alpha$.

**Step 2.** Establish the LESO by equation (9). One of the observer inputs is $u_a$, which can be easily measured by linear variable differential transformer (LVDT), and the other is $Q\ddot{x}_1 - m_a\ddot{x}_1 - m_a\ddot{z}_3$, where $z_3$ is the observer output.

**Step 3.** Adjust the observer gain $l_0$ to stabilise the controller.

Note that this stabilisation method is relevant to the observer gain $l_0$; therefore, it is necessary to analyse how $l_0$ affects the closed-loop system. The detailed analyses are presented in the following section.

### Stability and performance discussion

The accurate equations of the closed-loop system in $s$ domain considering the estimation error $e_3$ are written as follows

$$\begin{cases}
\begin{align*}
\left( (1 + \mu)s^2 + \Omega^2 \right) X_1(s) + \mu^2 U_a(s) = \frac{E_3(s)}{M} \\
\dot{s}^2 U_0(s) + s^2 X_1(s) = \frac{\mu^2}{\mu^2 + \alpha^2} sX_1(s) - 2\alpha s U_a(s) + E_3(s)
\end{align*}
\end{cases}$$

(19)

where $E_3(s)$ is the Laplace transformation of $e_3$. According to equations (11) and (13)

$$E_3(s) = \left( 1 - \frac{l_0^2}{(s + l_0)^2} \right) \left( -2(\xi\omega_a - \alpha)s - \omega_a^2 \right) U_a(s)$$

(20)

The characteristic equation of this system is written as follows

$$b_0 s^6 + b_1 s^5 + b_2 s^4 + b_3 s^3 + b_4 s^2 + b_5 s + b_6 = 0$$

(21)

where $b_0 = 1$, $b_1 = 3l_0 + J_1$, $b_2 = 3l_0^2 + 3J_1l_0 + J_2$, $b_3 = l_0^3 + 3J_1l_0^2 + 3J_2l_0 + J_3$, $b_4 = J_3l_0 + 3J_2l_0^2 + 3J_3l_0 + J_4$, $b_5 = \Omega^2 l_0^3 + 3J_3l_0^2 + 3J_3l_0$, and $b_6 = J_6l_0^3 + 3J_6l_0^2 + 3J_6l_0$. Moreover, $J_1 = 2(1 + \mu)\xi\omega_a + Q/M$, $J_2 = (1 + \mu)\omega_a^2 + \Omega^2$, $J_3 = 2\xi\omega_a\Omega^2$, $J_4 = 2(1 + \mu)\alpha + Q/M$, $J_5 = \Omega^2\omega_a^2$, and $J_6 = 2\alpha\Omega^2$.

To achieve the optimal passive vibration suppression, assume $\omega_a = \Omega$. Since $J_1$ and $J_4$ have a same form, assume that $\xi\omega_a = \alpha$ to simplify the analysis. Assuming that $\omega_a^2 = \Omega^2 = \eta_0$ and $J_1 = \lambda l_0$, where both $\eta$ and $\lambda$ are positive, $b_i, i = 0, 2, \ldots, 6$ can be simplified to be

$$\begin{cases}
\dot{b}_0 = 1, \dot{b}_1 = \beta_1 l_0 + \beta_2, \dot{b}_2 = \beta_3l_0^2 + \beta_4 l_0 + \beta_5, \\
\dot{b}_3 = \beta_6l_0^3 + \beta_7l_0^2 + \beta_8 l_0 + \beta_9, \dot{b}_4 = \lambda l_0^4 + 3\beta_3l_0^3 + \beta_8 l_0^2 \\
\dot{b}_5 = \eta_0^3 + \beta_7l_0^3, \dot{b}_6 = \beta_8 l_0^2
\end{cases}$$

(22)

where $\beta_1 = \lambda + 3$, $\beta_2 = 3\lambda + 3$, $\beta_3 = (2 + \mu)\eta$, $\beta_4 = (1 + 3\lambda)$, $\beta_5 = 2\alpha\eta$, $\beta_6 = \eta_0^2 + 6\alpha\eta$, $\beta_7 = 3\eta^2 + 6\alpha\eta$, and $\beta_8 = 3\eta^2 + 2\alpha\eta$.

According to the Routh–Hurwitz criterion, the sufficient and necessary condition to ensure the stability is that each element of the second column in Table 1 should be positive.

By calculating those elements with the help of Maple software, it can be seen that $c_{11} = \sum_{k=0}^{N_4} h_{ik} h_{ik}^{-k}$, $i = 1, 2, 3, 4$, where $h_{ik}$ and $N_i$ are the coefficient of $h_{ik}^{-k}$ and the polynomial degrees, respectively. In detail, $N_1 = 3$, $N_2 = 6$, $N_3 = 11$, and $N_4 = 18$. For example, $c_{11}$ is presented as follows

$$c_{11} = h_{10}l_0^3 + h_{11}l_0^2 + h_{12}l_0$$

(23)

where $h_{10} = 3\lambda^2 + 9\lambda + 8$, $h_{11} = (2 + \mu)\lambda\eta$, and $h_{12} = -2\alpha\eta$.

Since $h_{10} > 0$, $c_{11}$ can be positive, if $l_0$ is sufficiently large and far greater than other parameters ($\eta$, $\lambda$, and $\mu$). It follows that the rest $c_{ij}$, $i = 2, 3, 4$ can be positive as well under the same condition, because

**Table 1.** The Routh table.

<table>
<thead>
<tr>
<th>$s^6$</th>
<th>$b_{0}$</th>
<th>$b_{2}$</th>
<th>$b_{4}$</th>
<th>$b_{6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^5$</td>
<td>$b_{1}$</td>
<td>$b_{3}$</td>
<td>$b_{5}$</td>
<td>$b_{7}$</td>
</tr>
<tr>
<td>$s^4$</td>
<td>$c_{11} = b_{1}b_{2} - b_{0}b_{1}$</td>
<td>$c_{12} = b_{1}b_{4} - b_{0}b_{2}$</td>
<td>$b_{1}b_{6}$</td>
<td></td>
</tr>
<tr>
<td>$s^3$</td>
<td>$c_{13} = c_{12}b_{2} - c_{11}b_{1}$</td>
<td>$c_{14} = c_{12}b_{4} - c_{11}b_{2}$</td>
<td>$b_{1}b_{6}c_{12}$</td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td>$c_{15} = c_{14}b_{2} - c_{13}b_{1}$</td>
<td>$c_{16} = c_{14}b_{4} - c_{13}b_{2}$</td>
<td>$b_{1}b_{6}c_{12}$</td>
<td></td>
</tr>
<tr>
<td>$s$</td>
<td>$c_{17} = c_{16}b_{2} - c_{15}b_{1}$</td>
<td>$c_{18} = c_{16}b_{4} - c_{15}b_{2}$</td>
<td>$b_{1}b_{6}c_{12}$</td>
<td></td>
</tr>
</tbody>
</table>
Overall, for stabilisation, $l_0$ should be sufficiently large and far greater than $\eta, \alpha$, and $\mu$. This condition is easy to be satisfied for vibration control of flexible structures, because the values of both the structural natural frequency and the mass ratio are generally small. After stabilisation, the stability of the closed-loop system only requires a high value of $l_0$, which is feasible to be achieved by using a measurement system with a fast sampling frequency. Besides, $Q = M(l_0 - 2\alpha - 2\mu\alpha)$, which indicates that $Q$ is able to be as high as $l_0$ to ensure the effective control performance. Therefore, both the stability and the active control performance can be significantly increased after stabilisation.

Figure 2 shows the minimum $l_0$ to stabilise the controller for different natural frequencies of the primary structure. In this simulation, $M = 1, \mu = 0.03$, and $\xi \omega_0 = \alpha = 2$. To ensure the effective passive vibration suppression, the natural frequency of the DVA is tuned to be as same as the primary structure, that is, $\omega_p = \Omega$. Despite the increase of $Q$, the minimum $l_0$ for stabilisation are nearly same, which indicates that this stabilisation approach is robust to the control gain $Q$. In addition, the value of $l_0$ needs to be increased for stabilising the high-stiffness structure.

To show how the pole locations are influenced by the stabilisation approach, take the case where $\Omega = \omega_p = 14\pi$ as an example. Figure 3 shows the root locus of the system near the imaginary axis, which is developed by increasing the value of $Q$. If the system is not stabilised, the locus shows that a high value of $Q$ is unable to be chosen for effective control performance. In contrast, when $l_0 = 1000$ is utilised for stabilisation, all the poles and zeros are placed on open left-hand complex plane, which indicates that an arbitrary positive value of $Q$ can be chosen for active control. This can be qualitatively explained by equation (20) that $E(s)$ tends to zero, if $l_0 \to \infty$, thus the original unstable system is completely transformed to be the stable system shown in equation (15). However, for application, $l_0$ cannot be too large due to the restriction of both the sampling frequency and the noise of the measurement system. Besides, it is also unnecessary to use a too large value of $l_0$, because the stability improvement of the system can be achieved by a sufficient $l_0$ when the value of $Q$ has been determined (e.g. $l_0 > 100$ required for $Q = 800$ as shown in Figure 2). Therefore, an appropriately high value of $l_0$ is only required to accommodate the stability and performance in practice.

To validate the robustness and control performance subjected to significant parameter variations in the system, the following simulations based on the model shown in Figure 1 are conducted. The original structural parameters are given as follows: the mass of the primary structure $M$ is 1 kg, its natural frequency $\Omega$ is 7 Hz, a small damping of 0.1 N s/m is added into the primary structure to avoid the infinite resonant response; the mass of the DVA $m_b$ is 0.03 kg, its natural frequency $\omega_a$ is 7 Hz as well, and its damping $c_a$ is 0.12 N s/m. The control parameters are given as follows: $l_0 = 1000$ and $\alpha = 2$. According to equation (9), the parameter $b$ of the observer equals $1/m_a$. For the restriction of application all the control parameters should remain unchanged as far as is possible, when they have already been adjusted. Hence, all the control parameters are constant despite the parameter vibrations in the system.

Table 2 presents the maximum real components of the poles subjected to the structural parameter variations when $Q = 800$. Despite the marked variations, all the poles remain on open left-hand complex plane, which indicates that the present method can retain the stability subjected to the significant variations in both the primary
and the DVA. As is shown in Table 2, the controller is still stable, although the parameter $b$ for establishing the LESO does not equal $1/m_a$ due to the variations in $m_a$. This phenomenon can be explained qualitatively that the stabilisation approach actually transforms the hybrid DVA into the expected dynamic system (as shown in equation (14)) by eliminating the deviations between the two systems, thus the variations in $m_a$, treated as part of the derivation, can be eliminated as well. Since $m_a$ is easy to be measured, and its variations do not destabilise as compared with the natural frequency of the DVA, the value of $m_a$ is assumed to be constant in the following simulations.

Figure 4 presents frequency responses of the acceleration $x_1$ under stationary harmonic vibrations with unit amplitude. All the parameters remain unchanged in the case of Figure 4(a), whereas the natural frequency of the primary structure varies from 7 to 3.5 Hz in the case of Figure 4(b). The results in Figure 4(a) indicate that the resonant vibration is effectively suppressed by the hybrid DVA when it works as a passive absorber, while the global effectiveness is significantly improved by the present method in spite of the small degradation in a narrow band around 1 Hz. In addition, when $Q$ is increased, the performance is notably improved. In Figure 4(b), the passive vibration control becomes ineffective because the DVA is mistuned, when the natural frequency of the primary structure has changed far from its original value. However, the global active control performance is retained as well as the case in Figure 4(a).

### Multimode vibration control for comparison

For the sake of comparison, this section presents vibration control of a mass–spring layered structure, which is also used as an example in the study by Utsumi.\textsuperscript{13} The model is shown in Figure 5. The number of the layers is $N$, and the two hybrid DVAs are attached on the top layer in parallel with the axis of $x$ to suppress the translational vibration and the rotational vibration of the structure. The translational motion and the rotational motion of the $i$th layer are represented by $x_i$ and $\theta_i$, respectively. The dynamic equations of this system are

$$
\begin{align*}
\begin{cases}
M\ddot{x} + C_x \dot{x} + K_x x = -\delta(P_1 + P_2) \\
I \ddot{\theta} + C_{\theta} \dot{\theta} + K_{\theta} \theta = -\delta(P_2 \theta - P_1 \dot{\delta})
\end{cases}
\end{align*}
$$

where $\delta$ is a N-dimensional vector, which is $[0, 0, \ldots, 0, 1]^T$ in detail; $M$, $K_x$, and $C_x$ are the mass matrix, the translational stiffness matrix, and the translational damping matrix, respectively; $I$, $K_{\theta}$, and $C_{\theta}$ are the inertia matrix, the rotational stiffness matrix, and the rotational damping matrix, respectively. In particular, all these matrices are symmetric and positive definite. $K_i$ is given as an example

$$
K_i = \begin{bmatrix}
k_{i1} & k_{i2} & -k_{i2} & 0 & \cdots & 0 \\
0 & k_{i2} & k_{i3} & -k_{i3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-k_{iN} & 0 & 0 & \cdots & \cdots & k_{iN}
\end{bmatrix}_{N \times N}
$$

**Table 2.** Maximum real components of the systematic poles subjected to the parameter variations in both the primary structure and the DVA.

<table>
<thead>
<tr>
<th>Variation</th>
<th>$-50%$</th>
<th>$-25%$</th>
<th>$0$</th>
<th>$25%$</th>
<th>$50%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$-1.1822$</td>
<td>$-1.1758$</td>
<td>$-1.1695$</td>
<td>$-1.1633$</td>
<td>$-1.1570$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$0.2922$</td>
<td>$0.6573$</td>
<td>$1.1695$</td>
<td>$1.8299$</td>
<td>$2.6399$</td>
</tr>
<tr>
<td>$m_a$</td>
<td>$-2.3657$</td>
<td>$1.5657$</td>
<td>$-1.1695$</td>
<td>$0.9325$</td>
<td>$0.7747$</td>
</tr>
<tr>
<td>$\xi_a$</td>
<td>$-1.1624$</td>
<td>$-1.1659$</td>
<td>$-1.1695$</td>
<td>$-1.1731$</td>
<td>$-1.1767$</td>
</tr>
<tr>
<td>$\omega_a$</td>
<td>$-1.1927$</td>
<td>$-1.1830$</td>
<td>$-1.1695$</td>
<td>$-1.1524$</td>
<td>$-1.1319$</td>
</tr>
</tbody>
</table>

**Figure 4.** Frequency responses of $x_1$ for the cases of (a) no parameter variation and (b) the primary structural natural frequency decreased by 50%. DVA: dynamic vibration absorber.
where \( k_{ii} \) is the translational stiffness between the \( i \)th and \( j \)th layer. \( P_j \) is the reacting force generated by the \( j \)th DVA, which is written as follows

\[
P_j = m_{aj} \{ \ddot{u}_{aj} + [\ddot{x}_N + (-1)^j \ddot{\theta}_N] \} = f_{ij} - m_{aj} (2\xi_j \omega_j \dot{u}_{aj} + \omega_j^2 u_{aj})
\]

where \( m_{aj} \), \( \xi_j \), and \( \omega_j \) are the mass, the damping ratio, and the natural frequency of the \( j \)th DVA, respectively; \( u_{aj} \) is the relative motion between the DVA mass and its attachment.

The decentralised control strategy is applied, and the control law of the \( j \)th DVA is

\[
f_{ij} = Q_j [\ddot{x}_N + (-1)^j \ddot{\theta}_N] + f_{rj}
\]

where \( Q_j \) is the positive feedback gain and \( f_{rj} \) is the compensation force for stabilisation. Defining \( F_j = Q_j [\ddot{x}_N + (-1)^j \ddot{\theta}_N] - m_{aj} [\ddot{x}_N + (-1)^j \ddot{\theta}_N] + f_{rj} \), then, the LESO for the \( j \)th DVA is established to be

\[
\dot{z} = A \dot{z} + B F_j + L (u_{aj} - \hat{z}_3)
\]

where \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2\alpha_j & 1 \\ 0 & 0 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 1/m_{aj} \end{bmatrix} \), and

\[
L = \begin{bmatrix} 3l_0 - 2\alpha_j \\ 3l_0^2 - 6l_0 \alpha_j + 4\alpha_j^2 \\ l_0^3 \end{bmatrix}
\]

where \( l_0 > 0 \) and \( \alpha_j > 0 \). \( \hat{z}_3 \) is the estimation value of \( -2(\xi_j \omega_j \dot{u}_{aj} - \alpha_j) \dot{u}_{aj} - \omega_j^2 u_{aj} \). Choose a high value of \( l_0 \) to reduce the estimated error, and then let \( f_{rj} = -m_{aj} \dot{u}_{aj} \).

Therefore, equation (27) can be transformed into

\[
m_{aj} \{ \ddot{u}_{aj} + [\ddot{x}_N + (-1)^j \ddot{\theta}_N] \} = Q_j [\ddot{x}_N + (-1)^j \ddot{\theta}_N] - 2m_{aj} \alpha_j \dot{u}_{aj}
\]

(30)

The Lyapunov candidate function of the closed-loop system can be written as

\[
V = \frac{1}{2} \dot{x}^T M \dot{x} + \frac{1}{2} \dot{\theta}^T \Theta \dot{\theta} + \frac{1}{2} \ddot{\theta}^T \Theta \ddot{\theta}

+ \sum_{j=1}^2 \frac{m_{aj} \alpha_j [\dot{u}_{aj} + [\ddot{x}_N + (-1)^j \ddot{\theta}_N]]^2}{Q_j + 2m_{aj} \alpha_j}
\]

(31)

From equations (25) and (29), the derivative of \( V \) is

\[
\dot{V} = -\dot{x}^T C \dot{x} - \dot{\theta}^T \Theta \dot{\theta}

+ \sum_{j=1}^2 \frac{m_{aj} \alpha_j [\dot{u}_{aj} + [\ddot{x}_N + (-1)^j \ddot{\theta}_N]]^2}{Q_j + 2m_{aj} \alpha_j}
\]

(32)

As is shown in equation (32), \( \dot{V} < 0 \); thus, the closed-loop system is asymptotically stable.

For the sake of comparison, the structure parameters in the following simulations are as same as those in the study by Utsumi.\(^{13}\) The parameters of both the primary structure and the two hybrid DVAs are given in Table 3. A modal damping ratio 0.01 is introduced to each mode of the layered structure. An excitation force and an excitation moment are imposed on the top primary structure and the two hybrid DVAs are given meters in the following simulations are as same as those

Figure 5. The mass–spring layered structure used in the study by Utsumi\(^{13}\) (a) the profile and (b) two DVAs attached on the top layer. DVA: dynamic vibration absorber.
The present method still retains the effective performance as well as that shown in Figure 6 for the same case. It can also be seen that even though both the natural frequencies of the two DVAs have changed to be 50% of their original values, the stability and control performance can be still retained by the present method.

Figure 8 presents the control efforts of the second DVA by the two methods. The results of the previous method are only obtained for the case where there are no structural parameter variations. As is shown in Figure 8, the two methods have the same control efforts when none of the structural parameters varies. When both \( \omega_{ai} \) and \( \omega_{aj} \) are decreased by 50%, that is, from 0.5 to 0.25 Hz, the notch varies from 0.5 to 0.25 Hz as well, which can be interpreted as follows

\[
F_c(s) = (T_1(s) + T_2(s))[X_N(s) + l_2\theta_N(s)]
\]  

(a)  

![Control Efforts of the Second DVA](image)

(b)  

![Frequency Responses](image)

The parameters of the two hybrid DVAs

<table>
<thead>
<tr>
<th>DVA</th>
<th>( m_{dj} ) (kg)</th>
<th>( \xi_j )</th>
<th>( \omega_{dj} ) (rad/s)</th>
<th>( l_j ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>0.1</td>
<td>3.14</td>
<td>0.75</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>0.1</td>
<td>3.14</td>
<td>0.75</td>
</tr>
</tbody>
</table>

DVA: dynamic vibration absorber.

Table 3. The structural parameters used in Utsumi\(^{13}\).

<table>
<thead>
<tr>
<th>Stories (i)</th>
<th>Translational</th>
<th>Rotational</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( m_i ) (kg)</td>
<td>( k_{ti} ) (N/m)</td>
</tr>
<tr>
<td>1</td>
<td>354.2</td>
<td>159,100</td>
</tr>
<tr>
<td>2</td>
<td>354.2</td>
<td>93,270</td>
</tr>
<tr>
<td>3</td>
<td>354.2</td>
<td>50,350</td>
</tr>
<tr>
<td>4</td>
<td>354.2</td>
<td>50,350</td>
</tr>
<tr>
<td>5</td>
<td>354.2</td>
<td>24,280</td>
</tr>
<tr>
<td>6</td>
<td>393.6</td>
<td>24,280</td>
</tr>
</tbody>
</table>

Figure 6. Frequency responses of \( \ddot{x}_N + l_2\ddot{\theta}_N \) controlled by the two methods when none of the structural parameters varies.

Figure 7. Responses of \( \ddot{x}_N + l_2\ddot{\theta}_N \) subjected to the variations in the DVA natural frequencies: (a) time responses at 1.5 Hz controlled for the case where \( \omega_{ai} \), \( i = 1, 2 \) are decreased by 1% and (b) frequency responses of the present method for two variation situations.
where

\[
T_1(s) = \frac{Q(s^2 + 2\xi_2\omega_n s + \omega_n^2)}{s + 2\alpha_2},
\]

\[
T_2(s) = \frac{m_{a2}[2(\alpha_2 - \xi_2\omega_n) s^2 - \omega_n^2 s]}{s + 2\alpha_2}
\]

(34)

As is shown in equation (33), when the excitation frequency equals \(\omega_n\), a notch of the control efforts exists due to the small magnitude of \(T_1(s)\), thus the notch varies along with the variation in \(\omega_n\) in Figure 8. Since the hybrid DVA resembles a high-pass filter, the effort rises notably to generate a sufficient reacting force on the primary structure, when excitation frequency tends to zero. Generally, the excitation frequency is not very close to zero in application, thus the large control efforts are not necessary for active vibration control.

Consider another situation that the modal parameters markedly vary because of the change in the layer number, which is also presented in the study by Utsumi. In this case, the parameters of the controller are used as same as those in the case of Figure 6. As is shown in Figure 9, although the primary structure changes significantly, the present method still retains the stability and performance without changing any control parameters.

**Application in vibration control of a cantilever panel**

To validate the effectiveness of the present method, simulations on vibration control of a cantilever panel are conducted. The model is complicated to be derived analytically for the purpose to emulate the real industrial application, where sometimes, analytical models are not available for active control. Commercial dynamics software (NASTRAN and ADAMS) are adopted for modelling, where the panel is a finite element model (FEM) consisting of 240 elements and 785 nodes. The control algorithm is written in the MATLAB-Simulink. Based on ADAMS-Simulink co-simulation, the controller and the dynamic model are connected. Figure 10 shows the dynamic model built in ADAMS, where two hybrid absorbers are mounted at the end of the panel. The parameters of the panel are presented as follows: length, width, and thickness are 1, 0.6, and 0.005 m, respectively; the density is 2700 kg/m\(^3\); the Poisson ratio and the modulus are 0.3 and 69 GPa, respectively; and the modal damping ratio of each mode is 0.01. One side of the panel is fixed, whereas the others are free. Calculated by NASTRAN, the forms of first two resonances are vertical in the plane and torsional around the \(x\)-axis, and the resonant frequencies are 4.2 and 15.4 Hz, respectively. The parameters of the two absorbers are shown as follows

\[
m_{a1} = m_{a2} = 0.02 \text{ kg, } k_{a1} = 14 \text{ N/m, } k_{a2} = 187 \text{ N/m, } c_{a1} = 0.05 \text{ Ns/m, } c_{a2} = 0.3 \text{ Ns/m}
\]

(35)
where $m_{ai}$, $k_{ai}$, and $c_{ai}$ are the mass, stiffness, and damping of the $i$th absorber, respectively. The natural frequencies of the absorbers are close to the first and the second resonant frequencies of the cantilever panel, respectively, which indicates that the first two resonances can be effectively suppressed by passive vibration control.

Each absorber is stabilised by its own LESO, where the observer gain $l_0$ and the positive constant $\alpha$ are 1000 and 0.314, respectively. Both the control gains of skyhook damping $Q$ are 50. Figure 11(a) and (b) presents the power spectrum density (PSD) of point (1, 0) and point (1, 0.6), when a random force, emulated by a white noise signal with the PSD of $1 \times 10^{-6}$ dB/Hz, is imposed on point (0.5, 0). In the simulation, the sampling frequency is 1000 Hz. Results show that the first two resonances are suppressed when both absorbers work in the passive control mode, while the global control performance is significantly improved when the present method is adopted for active control. Besides, the maximum control forces are 0.0335 and 0.0395 N, respectively, which indicates the low power consumption.

Consider that the stiffness of each absorber changes notably, where the stiffness of the first absorber decreases by 50% and the other increases by 50%. Without changing any control parameter, the PSD of point (1, 0) is shown in Figure 12, which indicates that the active control is still effective and stable in spite of the large parameter variations, although passive control becomes ineffective here due to the mistuning of the absorbers. Therefore, the performance of this active control method for vibration control of complex structures is demonstrated.

**Conclusion**

This article has proposed a robust control method of an active–passive hybrid vibration absorber for vibration suppression in a wide frequency band based on skyhook damping strategy. Theoretical analysis indicates that a high control gain for effective control performance can be utilised in the skyhook damping loop when it is stabilised by the technology of LESO.

The SDOF primary structure with a hybrid vibration absorber is presented to discuss the stability and performance of the present method. The stability of the closed-loop system is significantly improved by increasing the observer gain. According to the pole location analysis, the system controlled by the present method becomes strongly robust against marked parameter variations in both the primary structure and the vibration absorber after stabilisation. Compared with passive vibration control, the present method achieves more effectiveness in the global vibration control in spite of a small degradation in a narrow low-frequency band.

When the method is extended to control multiple hybrid vibration absorbers for multimode vibration
suppression, decentralised control strategy can be employed, which is demonstrated to be asymptotically stable by the Lyapunov stability criterion. Therefore, the design procedure of the controller is simple. Numerical simulation results show that the stability can be ensured regardless of the marked variations in the natural frequency of the hybrid vibration absorber, whereas the previous approach becomes unstable due to minor changes in this parameter. In addition, the robustness against the variation in the primary structural modes is also validated, and all the vibration modes are effectively suppressed without changing any control parameters. Finally, vibration control of a cantilever panel by means of co-simulation of several commercial software is studied to validate the effectiveness of the method when it is applied for complex multimode structures, where analytical models are not available. Simulation results prove that random vibrations are significantly suppressed in spite of the large changes of each absorber’s stiffness. Hence, both the robustness and the control performance of the present method for multimode vibration control are demonstrated.

**Funding**

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**References**


**Appendix 1**

**Notation**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>transfer matrix of linear extended state observer (LESO)</td>
</tr>
<tr>
<td>B</td>
<td>input matrix of LESO</td>
</tr>
<tr>
<td>C_{a1}, C_{a2}</td>
<td>damping constants of hybrid absorbers</td>
</tr>
<tr>
<td>C_{r}, C_{t}</td>
<td>rotational and translational damping matrices of the layered structure</td>
</tr>
<tr>
<td>d</td>
<td>perturbation</td>
</tr>
<tr>
<td>e</td>
<td>vector of estimation errors</td>
</tr>
<tr>
<td>E</td>
<td>input matrix of perturbation control forces</td>
</tr>
<tr>
<td>f_{c}, f_{A1}, f_{A2}</td>
<td>control forces</td>
</tr>
</tbody>
</table>
\( f_d \) disturbance force

\( f_r, f_{1r}, f_{2r} \) compensation forces

\( I, M \) inertia and mass matrices of the layered structure

\( k_a, k_{a1}, k_{a2} \) stiffness constants of hybrid absorbers

\( K_r, K_t \) rotational and translational stiffness matrices of the layered structure

\( l_0 \) observer gain of LESO

\( l_1, l_2 \) positions of hybrid absorbers

\( L \) gain vector of LESO

\( m_a, m_{a1}, m_{a2} \) masses of hybrid absorbers

\( M \) mass of the single-degree-of-freedom (SDOF) primary structure

\( P, P_1, P_2 \) reacting forces of hybrid absorbers

\( Q, Q_1, Q_2 \) control gains of skyhook damping

\( u_a, u_{a1}, u_{a2} \) relative motions of masses of absorbers

\( V \) Lyapunov candidate function

\( x_1 \) translational motion of the SDOF structure

\( x \) translational motion vectors of the layered structure

\( \hat{z} \) estimated value of perturbation

\( \hat{\theta} \) vector of values estimated by LESO

\( a, a_1, a_2 \) positive constants in LESO

\( \alpha, \alpha_1, \alpha_2 \) N-dimensional vector

\( \mu \) mass ratio between absorber and the SDOF structure

\( \theta \) rotational motion vectors of the layered structure

\( \omega \) excitation frequency

\( \omega_a, \omega_{a1}, \omega_{a2} \) natural frequencies of hybrid absorbers

\( \Omega \) natural frequency of the SDOF primary structure

\( \xi, \xi_1, \xi_2 \) damping ratios of hybrid absorbers

Yang et al.